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An Optimal Two-Warehouse Inventory Model for Deteriorating Items with Quadratic Demand Under Downstream Trade Credit

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Abstract

In inventory management, one of the facility used to stimulate the demand of customer is trade credit. Thus, trade credit has been explored in the upstream level and concurrrntly at both upstream and downstream levels when the demand is quadratic in nature. However, it has not been explored in the downstream level. In this study, a trade credit has been allowed at downstream level only in a two-warehouse invenotory manmagement policy where the demand is assumed to be quadratic. This is to stimulate the demand of customers when the retailer is involved in large orders. The cost functions of the model were established using the solution of the differential equations that represents the inventory level's behavior for different levels. As an illustration of the model, a numerical example was given and it revealed that the goods at the rented warehouse finishes at $t_z^* =$1.2850 while the replenishment cycle length ends at $T^* =$5.0739 that yielded the optimal cost of $TC^* = 4094.325$. The result of the sensitivity analysis carried out shows that, the retailer should keep less quantity of goods in own warehouse (OW) and the larger quantity at the rented warehouse (RW) to get a higher profit because of the preserving facilities in the RW that reduces the deterioration effect on the items. This model is applicable to newly introduced goods in the market especially fashionable items that their demand grew and start to decline with time.

Keywords: Deterioration items; Two-warehouse Inventory Model; Inventory management; Trade credit; Qudratic demand; Downstream trade credit

Introduction

For decades, inventory management of deteriorating items have been studied widely in capacity constrained environments considering different sales driven factors such as trade credit. In those studies, the trade credit was allowed at different levels including upstream and both upstream and downstream. But, the trade credit has not been explored in the downstream only. This has give room for further conisderation of the downstream trade credit only in the two-warehouse environments.

In the classical studies, inventory models were developed considering single warehouse. The warehouse is presumed to be of unlimited stocking capacity. However, this assumption may not represent some situations where the warehouse might turn out to be capacity constrained especially when the retailer is involved in placing large orders. This suggest that, in case of large order, the warehouse may be filled up and the need to have another warehouse known as rented warehouse (RW) to keep the excess is necessary. One of the factors that the retailer might consider to place large orders is when s/he is given the grace of trade credit by the supplier (Upstream). Liang and Zhou (2011) developed inventory model when the trade credit is only allowed at upstream level. The items conisdered are instantaneoulsy deteriorating. For more on studies that considered only upstream in a two-warehouse environment, see Bhunia et al (2014); Aliyu (2025); Aliyu and Ibrahim (2025); Haruna Aliyu and Adamu (2025); and Oluwafemi and Aliyu (2026).

To stimulate the demand of customer to reduce the high holding cost associated with the RW, the

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retailer extended the grace of permissible delay in payment to his/her customers known as two-level trade credit (upstream and downstream). In that regard, Aliyu (2022) developed the inventory model for deteriorating items when the trade credit is allowed at both upstream and downstream. Aliyu and Sani (2021) allowed the downstream at both upstream and downstream to investigate the credit riskiness of retailer. For more on studies that conisdered both upstream and downstream in a two-warehouse environment, see Aliyu and Sani (2020); Aliyu and Sani (2022); and Aliyu and Gwafan (2026).

Since the introduction of sales driven upstream trade credit by Goyal (1985) and its subsequent extension by Huang (2003) to consider both upstream and downstream trade credit, all inventory management models that considers trade credit followed same pattern except Malumfashi et al (2023) that considers the trade credit at downstream level only. In that model, they consider demand to be time varying and the warehouse to be single with unlimited capacity. However, no study has examined downstream trade credit within a capacity constrained warehouse setting, which creates a clear gap in the literature.

Another important factor in the management of inventory is the demand. In some studies such as Aliyu and Sani (2021), the demand was assumed to be constant which is not always the case. That motivate some others to consider situation when the demand is assumed linearly dependent on time. Malumfashi et al (2023) assumed the time to be linearly increasing with time and that the production exceeds the demand. Aliyu and Gwafan (2026) assumed the demand to be rapid due to the trade credit given to the customers. However, for some stylish or fashionable items, their demand respond to time, although it can be increasing or rapidly increasing, it start to decrease with time which makes it quadratic.

In this study we are motivated to extend the idea of downstream trade credit in Malumfashi et al. (2023) to a two-warehouse environment. The model is for deteriorating items whose demand is assumed to be quadratic where the two warehouses have different holding costs because they possessed different preservation facilities.

Therefore, The model considers the joint impact of demand variation and credit incentives on inventory performance, providing useful insights for decision makers in manufacturing and retail sectors, and a two-warehouse inventory model for deteriorating items having quadratic demand under downstream trade credit would be developed.

The remaining sections are as follows: in 2, the model was formulated. Section 3 presents the optimization and analysis of the model whereas in 4, the numerical example was given. The conclusion was given in section 5.

Mathematical Model Formulation

Notations and Assumptions

The following are the parameters used in the model

i. $\theta_r(t), \theta_0(t)$ are the inventory levels of the rented warehouse (RW) and own warehouse (OW) respectively at time t.
ii. t_z and T are the times at which inventory in RW and OW drops to zero respectively.
iii. $D(t)$ is the time dependent demand rate which is $D(t) = (at^2 + bt + c)$, a, b, $c > 0$,
iv. Z is the maximum quantity that can be stored in OW.
v. h_r, h_o are the holding costs per unit time of RW and OW respectively.
vi. μ, λ are the deterioration rates in OW and RW respectively, with $\mu > \lambda$
vii. N is the trade credit period offered to the customers by the retailer
viii. I_c, I_e are the interest charged and interest earned return rates respectively
ix. $C_{1,} C_{2,} C_3$ are the ordering cost, purchasing cost and selling price respectively.
x. TC is the annual total cost of the model to be minimized.

In building the model, the following assumptions were made:

(i). The supplier does not give trade credit grace to the retailer for the items ordered.
(ii). The retailer give trade credit period to the customers.
(iii). Demand rate is assumed to be quadratic that is $D(t)=at^2+bt+c$ $(a,b,c>0)$.
(iv). The deterioration rate is constant.
(v). The inventory system involves single type of items.
(vi). Shortage is not allowed and lead time is zero.

Model formulation

At t =0, the total stocks are received and after filling OW to its capacity, the excess was stored in RW. The stocks are first dispatched from RW and at $t = t_z$ the inventory at RW drops to zero due to demand and deterioration during the period [0, t_z], while items in own warehouse (OW) reduces as a result of deterioration only. Now, both warehouses become empty at $t = T$ as a result of depletion due to demand and deterioration during the period [t_z,T]. These phenomenon as depcited in Fig 1 are represented by the following differential equations:

$$\frac{d\theta_r(t)}{dt} + \mu\theta_r(t) = -(at^2+bt+c) \qquad 0 \le t \le t_z \tag{1}$$

with boundary condition $\theta_r(t_z) = 0$

$$\frac{d\theta_0(t)}{dt} + \lambda\theta_0(t) = 0 \qquad 0 \le t \le t_z \tag{2}$$

with initial condition $\theta_0(0) = Z$

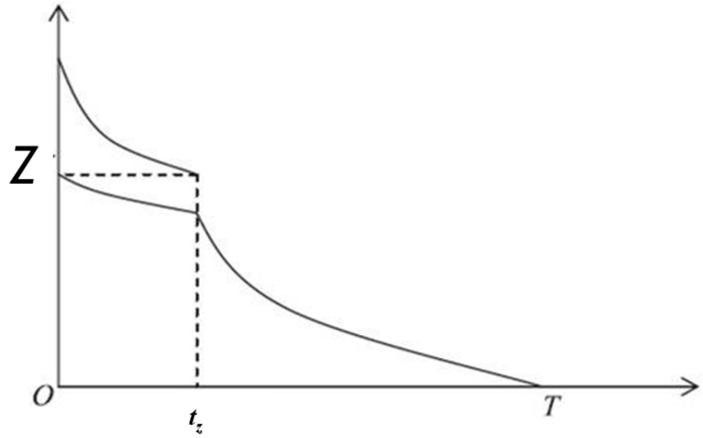

Figure 1: Diagramatical Representation of the Model

and

$$\frac{d\theta_0(t)}{dt} + \lambda\theta_0(t) = -(at^2+b\,t+c) \qquad t_z \le t \le T \tag{3}$$

with boundary condition $\theta_0(T) = 0$.

Solving eqns (1), (2) and (3) using the initial and boundary conditions, we get the respective solutions as:

$$\theta_r(t) = -\frac{1}{\mu}\left\{\left\{(at^2+bt+c) - \frac{(2at+b)}{\mu} + \frac{2a}{\mu^2}\right\} - e^{-\mu(t_z-t)}\left\{at_z^2 + bt_z + c - \frac{(2at_z+b)}{\mu} + \frac{2a}{\mu^2}\right\}\right\} \tag{4}$$

$$\theta_0(t) = Ze^{-\lambda t} \tag{5}$$

$$\theta_0(t) = -\frac{1}{\lambda}\left\{\left\{(at^2+bt+c) - \frac{(2at+b)}{\lambda} + \frac{2a}{\lambda^2}\right\} - e^{-\lambda(T-t)}\left\{aT^2 + bT + c - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2}\right\}\right\} \tag{6}$$

For the model to stand, there must be continuity in OW at $t = t_z$. Hence substituting $t = t_z$ into (5) and (6) and equating the resultant functions, after simplifying, we get

$$Z = -\frac{1}{\lambda}\left\{\left\{(at_z^2+bt_z+c) - \frac{(2at_z+b)}{\lambda} + \frac{2a}{\lambda^2}\right\} - e^{-\lambda(T-t_z)}\left\{aT^2 + bT + c - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2}\right\}\right\}e^{\lambda t_z} \tag{7}$$

Annual Total Costs

The total costs in this model are the Ordering Cost (OC), Holding Cost (HC), Deterioration Cost (DC), Interest Earned (EI), and Interest payable (IP). Therefore, the annual total costs (TC) is given by

$$TC = OC + HC + DC - IE + IP \tag{8}$$

Annual ordering cost (OC)

The annual ordering cost is given as

$$OC = \frac{A}{T} \tag{9}$$

Annual holding cost (HC)

The cost of holding the goods for the period (0, T) is $\frac{h_r}{T}\int_0^{t_z}\theta_r(t)dt + \frac{h_0}{T}\left\{\int_0^{t_z}\theta_0(t)dt + \int_{t_z}^{T}\theta_0(t)dt\right\}$

Therefore, using equations (4), (5) and (6), and after little algebraic simplification, the total annual holding cost for both RW and OW, is given by

$$HC = -\frac{h_r}{\mu T}\left\{\left(\frac{at_z^3}{3} + \frac{bt_z^2}{2} + ct_z - \frac{(at_z^2+bt_z)}{\mu} + \frac{2at_z}{\mu^2} + \frac{1}{\mu}\left(at_z^2 + bt_z + c - \frac{(2at_z+b)}{\mu} + \frac{2a}{\mu^2}\right) - \frac{e^{\mu t_z}}{\mu}(at_z^2 + bt_z + c - \frac{(2at_z+b)}{\mu} + \frac{2a}{\mu^2})\right\} + \frac{Zh_0}{\lambda T}\left\{1 - e^{-\lambda t_z}\right\} - \frac{h_0}{\lambda T}\left\{\frac{aT^3}{3} + \frac{bT^2}{2} + cT - \frac{(aT^2+bT)}{\lambda} + \frac{2aT}{\lambda^2} + \frac{1}{\lambda}\left(aT^2 + bT + c - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2}\right) - \left(\frac{at_z^3}{3} + \frac{bt_z^2}{2} + ct_z - \frac{(at_z^2+bt_z)}{\lambda} + \frac{2at_z}{\lambda^2} + \frac{e^{\lambda(T-t_z)}}{\lambda}\left(aT^2 + bT + C - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2}\right)\right)\right\} \quad (10)$$

Annual deterioration cost (DC)

The annual deterioration cost is given as

$$DC = \frac{C_3}{T}\{\mu\int_0^{t_z}\theta_r(t)dt + \lambda\int_0^{t_z}\theta_0(t)dt + \lambda\int_{t_z}^{T}\theta_0(t)dt\}$$

Therefore, using equations (4), (5) and (6), evaluating the integrals and simplifying, the total annual deterioration cost is given by

$$DC = -\frac{C_3}{T}\left\{\frac{at_z^3}{3} + \frac{bt_z^2}{2} + ct_z - \frac{(at_z^2+bt_z)}{\mu} + \frac{2at_z}{\mu^2} + \frac{1}{\mu}\left(at_z^2 + bt_z + c - \frac{(2at_z+b)}{\mu} + \frac{2a}{\mu^2}\right) - \frac{e^{\mu t_z}}{\mu}\left(at_z^2 + bt_z + c - \frac{(2at_z+b)}{\mu} + \frac{2a}{\mu^2}\right)\right\} + \frac{ZC_3}{T}\left\{1 - e^{-\lambda tz}\right\} - \frac{C_3}{T}\left\{\frac{aT^3}{3} + \frac{bT^2}{2} + cT - \frac{(aT^2+bT)}{\lambda} + \frac{2aT}{\lambda^2} + \frac{1}{\lambda}\left(aT^2 + bT + c - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2} - \left(\frac{at_z^3}{3} + \frac{bt_z^2}{2} + ct_z - \frac{(at_z^2+bt_z)}{\lambda} + \frac{2at_z}{\lambda^2}\right) + \frac{e^{\lambda(T-t_z)}}{\lambda}\left(aT^2 + bT + c - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2}\right)\right)\right\} \quad (11)$$

Annual interest payable, I_P and annual interest earned, I_E

The annual interest paid, and annual interest earned is based on the values and positions of N, t_z and T as can be seen in Fig 2. From the assumption, $N < t_z < T$ is restricted and therefore only one case is possible.

Case: $N < t_z < T$ (Trade credit period given to the customers is less than the replenishment)

Now, since supplier does not offer trade credit to the retailer; hence, the annual total interest payable by the retailer is

$$I_P = 0 \quad (12)$$

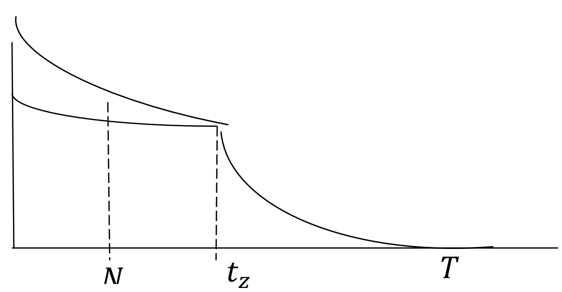

Figure 2: Representation showing only possible position of N.

Since there is still unsettled amount for the unsold goods with the customer at the expiration of the trade credit period given, the retailer will earn interest in the sales revenue recovered from the customer during the period $[N, T]$ in two ways (scenarios).

SCENARIO 1; (When partial payment is accepted by the retailer to be made at $t = N$ and the balance at the time after $t = N$)

Supposed that the retailer accepts partial payment at $t = N$ and will balance the remaining amount $C_3 - C_R$ after N i.e. $t = \alpha$. Now the total amount paid by customer, let's say C_R during the period $[0, N]$ is

$$C_R = C_3\int_0^N D(t)dt = C_3\int_0^N (at^2+bt+c)\,dt = C_3(\frac{aN^3}{3}+\frac{bN^2}{2}+cN) \quad (13)$$

We assume the rest amount to be paid to the retailer at $t=\alpha$ to be C_3Z-C_R, where $\alpha > N$.
Therefore, the total annual interest payable by the customer to the retailer at time $t=\alpha$ for scenario 1, let's say l_{E_1} is given as

$$l_{E_{11}} = \frac{I_e}{T}\int_N^\alpha \left(C_3\theta_0(t)e^{\lambda t} - C_R\right)dt = \frac{I_e}{T}\left\{\frac{C_3\theta_0(t)\left(e^{\lambda N}-e^{\lambda\alpha}\right)}{\lambda} - C_R(\alpha-N)\right\} = \frac{I_eC_3}{T}\left\{\frac{\theta_0(t)\left(e^{\lambda\alpha}-e^{\lambda N}\right)}{\lambda} - \left(\frac{aN^3}{3}+\frac{bN^2}{2}+cN\right)\left((\alpha-N)\right)\right\} \quad (14)$$

Therefore, the total annual relevant costs for this case are obtained by putting (9), (10) (11), (12), and (14) into equation (8), we have

$$TC_{11} = \frac{1}{T}\left\{A + \frac{h_r}{\mu}\left\{(\frac{at_z^3}{3}+\frac{bt_z^2}{2}+ct_z - \frac{(at_z^2+bt_z)}{\mu}+\frac{2at_z}{\mu^2}+\frac{1}{\mu}\left(at_z^2+bt_z+c-\frac{(2at_z+b)}{\mu}+\frac{2a}{\mu^2}\right) - \frac{e^{\mu t_z}}{\mu}(at_z^2+bt_z+c-\frac{(2at_z+b)}{\mu}+\frac{2a}{\mu^2})\right\} + \frac{Zh_0}{\lambda}\{1-e^{-\lambda t_z}\} + \frac{h_0}{\lambda}\left\{\frac{aT^3}{3}+\frac{bT^2}{2}+cT-\frac{(aT^2+bT)}{\lambda}+\frac{2aT}{\lambda^2}+\frac{1}{\lambda}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right) - (\frac{at_z^3}{3}+\frac{bt_z^2}{2}+ct_z-\frac{(at_z^2+bt_z)}{\lambda}+\frac{2at_z}{\lambda^2}+\frac{e^{\lambda(T-t_z)}}{\lambda}\left(aT^2+bT+C-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right))\right\} + C_3\left\{\frac{at_z^3}{3}+\frac{bt_z^2}{2}+ct_z-\frac{(at_z^2+bt_z)}{\mu}+\frac{2at_z}{\mu^2}+\frac{1}{\mu}\left(at_z^2+bt_z+c-\frac{(2at_z+b)}{\mu}+\frac{2a}{\mu^2}\right)-\frac{e^{\mu t_z}}{\mu}\left(at_z^2+bt_z+c-\frac{(2at_z+b)}{\mu}+\frac{2a}{\mu^2}\right)\right\} + ZC_3\{1-e^{-\lambda t_z}\} + C_3\left\{\frac{aT^3}{3}+\frac{bT^2}{2}+cT-\frac{(aT^2+bT)}{\lambda}+\frac{2aT}{\lambda^2}+\frac{1}{\lambda}\,(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}-(\frac{at_z^3}{3}+\frac{bt_z^2}{2}+ct_z-\frac{(at_z^2+bt_z)}{\lambda}+\frac{2at_z}{\lambda^2})+\frac{e^{\lambda(T-t_z)}}{\lambda}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)\right\} - I_eC_3\left\{\frac{\theta_0(t)\left(e^{\lambda\alpha}-e^{\lambda N}\right)}{\lambda}-\left(\frac{aN^3}{3}+\frac{bN^2}{2}+cN\right)\left((\alpha-N)\right)\right\}\right\} \quad (15)$$

Scenario 2: (It is assumed that partial payment is not acceptable at $t>N$)

This scenario means that full payment is to be made after $t=N$ due to the retailer unwillingness to accept partial payment. Hence, let full payment be made at time, $t=\beta$ where $\beta > N$.
Nevertheless, in this scenario the interest payable by customer let say $l_{E_{12}}$ to the retailer is given as

$$l_{E_{12}} = \frac{I_e}{T}\int_N^\beta \left(C_3\theta_0(t)e^{\lambda t}\right)dt = \frac{I_eC_3}{T\lambda}\{\theta_0(t)\left(e^{\lambda\beta} - e^{\lambda N}\right)\} \quad (16)$$

Where $Z = \theta_0(t)e^{\lambda t}$ and $t=\beta$
Therefore, the total annual relevant costs for scenario 2 is obtained by putting (9), (10) (11), (12), and (16) into equation (8) and we have that scenario 2

$$TC_{12} = \frac{1}{T}\left\{A + \frac{h_r}{\mu}\left\{(\frac{at_z^3}{3}+\frac{bt_z^2}{2}+ct_z - \frac{(at_z^2+bt_z)}{\mu}+\frac{2at_z}{\mu^2}+\frac{1}{\mu}\left(at_z^2+bt_z+c-\frac{(2at_z+b)}{\mu}+\frac{2a}{\mu^2}\right) - \frac{e^{\mu t_z}}{\mu}(at_z^2+bt_z+c-\frac{(2at_z+b)}{\mu}+\frac{2a}{\mu^2})\right\} + \frac{Zh_0}{\lambda}\{1-e^{-\lambda t_z}\} + \frac{h_0}{\lambda}\left\{\frac{aT^3}{3}+\frac{bT^2}{2}+cT-\frac{(aT^2+bT)}{\lambda}+\frac{2aT}{\lambda^2}+\frac{1}{\lambda}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right) - (\frac{at_z^3}{3}+\frac{bt_z^2}{2}+ct_z-\frac{(at_z^2+bt_z)}{\lambda}+\frac{2at_z}{\lambda^2}+\frac{e^{\lambda(T-t_z)}}{\lambda}\left(aT^2+bT+C-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right))\right\} + C_3\left\{\frac{at_z^3}{3}+\frac{bt_z^2}{2}+ct_z-\frac{(at_z^2+bt_z)}{\mu}+\frac{2at_z}{\mu^2}+\frac{1}{\mu}\left(at_z^2+bt_z+c-\frac{(2at_z+b)}{\mu}+\frac{2a}{\mu^2}\right)-\frac{e^{\mu t_z}}{\mu}\left(at_z^2+bt_z+c-\frac{(2at_z+b)}{\mu}+\frac{2a}{\mu^2}\right)\right\} + ZC_3\{1-e^{-\lambda t_z}\} + C_3\left\{\frac{aT^3}{3}+\frac{bT^2}{2}+cT-\frac{(aT^2+bT)}{\lambda}+\frac{2aT}{\lambda^2}+\frac{1}{\lambda}\,(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}-(\frac{at_z^3}{3}+\frac{bt_z^2}{2}+ct_z-\frac{(at_z^2+bt_z)}{\lambda}+\frac{2at_z}{\lambda^2})+\frac{e^{\lambda(T-t_z)}}{\lambda}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)\right\} - \frac{I_eC_3}{\lambda}\{\theta_0(t)\left(e^{\lambda\beta} - e^{\lambda N}\right)\}\right\} \quad (17)$$

Optimality Conditions and Analysis

The objective of this study is to find the optimum cost of maintaining the inventory, by minimizing the total cost functions TC_{11} hence, the necessary conditions to obtain a minimum are $\frac{\partial TC_{11}}{\partial t_z} = 0$ and $\frac{\partial TC_{11}}{\partial T} = 0$.

Using equation (15), by differentiating partially with respect to t_z and T respectively and setting the derivatives to zero. Differentiating w.r.t t_z

$$\frac{\partial TC_{11}}{\partial t_z} = -\frac{1}{T}\Bigg\{\frac{h_r}{\mu}\left(at_z^2+bt_z+c+e^{\mu t_z}\left(at_z^2+bt_z+c-\frac{(2at_z+b)}{\mu}+\frac{2a}{\mu^2}\right)+\frac{e^{\mu t_z}}{\mu}\left(2at_z+b-\frac{2a}{\mu}\right)\right)+ \{Zh_0e^{-\lambda t_z}\}+\frac{h_0}{\lambda}\left\{at_z^2+bt_z+c-\frac{(2at_z+b)}{\lambda}+\frac{2a}{\lambda^2}-e^{\lambda(T-t_z)}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)\right\}+C_3\left(at_z^2+bt_z+ c+e^{\mu t_z}\left(at_z^2+bt_z+c-\frac{(2at_z+b)}{\mu}+\frac{2a}{\mu^2}\right)+\frac{e^{\mu t_z}}{\mu}\left(2at_z+b-\frac{2a}{\mu}\right)\right)+\{ZC_3\lambda e^{-\lambda t_z}\}+C_3\left\{at_z^2+bt_z+c- \frac{(2at_z+b)}{\lambda}+\frac{2a}{\lambda^2}-e^{\lambda(T-t_z)}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)\right\}\Bigg\}=0 \quad (18)$$

Differentiating equation (15) w.r.t T and setting the result to zero

$$\frac{\partial TC_{11}}{\partial T} = -\frac{1}{T}\Bigg\{\frac{h_o}{\lambda}\left(aT^2+bT+c-\frac{2a}{\lambda}+\frac{2a}{\lambda^2}-e^{\lambda(T-t_z)}\left(aT^2+bT+c-\left(\frac{2aT+b}{\lambda}\right)+\frac{2a}{\lambda^2}+\frac{e^{\lambda(T-t_z)}}{\lambda}\left(2aT+b- \frac{2a}{\lambda}\right)\right)\right)+C_3\left(aT^2+bT+c-\frac{2a}{\lambda}+\frac{2a}{\lambda^2}-e^{\lambda(T-t_z)}\left(aT^2+bT+c-\left(\frac{2aT+b}{\lambda}\right)+\frac{2a}{\lambda^2}+\frac{e^{\lambda(T-t_z)}}{\lambda}\left(2aT+b- \frac{2a}{\lambda}\right)\right)\right)-TC_{11}\Bigg\}=0 \quad (19)$$

Solving the nonlinear equations (18) and (19), would give the values of t_z^* and T^*. To confirm they exist and unique, we show that $\frac{\partial^2 TC_{11}}{\partial t_z^2}\frac{\partial^2 TC_{11}}{\partial T^2}-\frac{\partial^2 TC_{11}}{\partial t_z\partial T}\frac{\partial^2 TC_{11}}{\partial T\partial t_z}>0$ at (t_z^*,T^*).

Therefore, differentiating equation (18) w.r.t t_z, we have

$$\frac{\partial^2 TC_{11}}{\partial t_z^2} = \frac{1}{T}\Bigg\{\frac{h_r}{\mu}\left(2at_z+b+\mu e^{\mu t_z}\left(at_z^2+bt_z+c-\frac{(2at_z+b)}{\mu}+\frac{2a}{\mu^2}\right)-2e^{\mu t_z}\left(2at_z+b-\frac{2a}{\mu}\right)-2a\frac{e^{\mu t_z}}{\mu}\right)+ \{Zh_0\lambda e^{-\lambda t_z}\}+\frac{h_0}{\lambda}\left\{2at_z+b-\frac{2a}{\lambda}+\lambda e^{\lambda(T-t_z)}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)\right\}+C_3\left(2at_z+b-\mu e^{\mu t_z}\left(at_z^2+ bt_z+c-\frac{(2at_z+b)}{\mu}+\frac{2a}{\mu^2}\right)-2e^{\mu t_z}\left(2at_z+b-\frac{2a}{\mu}\right)-2a\frac{e^{\mu t_z}}{\mu}\right)+\{ZC_3\lambda^2e^{-\lambda t_z}\}+C_3\left\{2at_z+b-\frac{2a}{\lambda}+ \lambda e^{\lambda(T-t_z)}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)\right\}\Bigg\} \quad (20)$$

Differentiating equation (19) w.r.t T, we have

$$\frac{\partial^2 TC_{11}}{\partial T^2} = \frac{1}{T}\Bigg\{\frac{h_o}{\lambda}\left\{\left(2aT+b-\lambda e^{\lambda(T-t_z)}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)-e^{\lambda(T-t_z)}(2aT+b)+e^{\lambda(T-t_z)}(2aT+ b)+\frac{2ae^{\lambda(T-t_z)}}{\lambda}\right)\right\}+C_3\left\{\left(2aT+b-\lambda e^{\lambda(T-t_z)}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)-e^{\lambda(T-t_z)}(2aT+b)+ e^{\lambda(T-t_z)}(2aT+b)+\frac{2ae^{\lambda(T-t_z)}}{\lambda}\right)\right\}-2\frac{\partial TC_{11}}{\partial T}\Bigg\} \quad (21)$$

By theorem 1:

Since $\left.\frac{\partial TC_{11}}{\partial T}\right|_{(t_z^*,T^*).}=0$. Thus, value of $\left.\frac{\partial^2 TC_{11}}{\partial T^2}\right|_{(t_z^*,T^*).}>0$,

Differentiating the left hand side of equation (19) w.r.t t_z, we have

$$\frac{\partial^2 TC_{11}}{\partial T\partial t_z} = -\frac{1}{T}\left\{h_o\left(e^{\lambda(T-t_z)}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)+e^{\lambda(T-t_z)}\left(2aT+b-\frac{2a}{\lambda}\right)\right)+C_3\left(e^{\lambda(T-t_z)}\left(aT^2+ bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)+e^{\lambda(T-t_z)}\left(2aT+b-\frac{2a}{\lambda}\right)\right)+\frac{\partial TC_{11}}{\partial t_z}\right\} \quad (22)$$

Also, differentiating the left hand side of equation (18) w.r.t T, we have

$$\frac{\partial^2 TC_{11}}{\partial t_z \partial T} = -\frac{1}{T}\left\{h_o\left(e^{\lambda(T-t_z)}\left(aT^2 + bT + c - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2}\right) + e^{\lambda(T-t_z)}\left(2aT + b - \frac{2a}{\lambda}\right)\right) + C_3\left(e^{\lambda(T-t_z)}\left(aT^2 + bT + c - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2}\right) + e^{\lambda(T-t_z)}\left(2aT + b - \frac{2a}{\lambda}\right)\right) + \frac{\partial TC_{11}}{\partial t_z}\right\} \quad (23)$$

Evaluating equations (22) and (23) at the point (t_z^*, T^*) and noting that $\left.\frac{\partial TC_{11}}{\partial T}\right|_{(t_z^*,T^*).} = 0$

$$\left.\frac{\partial^2 TC_{11}}{\partial T \partial t_z}\right|_{(t_z^*,T^*).} = -\frac{1}{T}h_o\left(e^{\lambda(T-t_z)}\left(aT^2 + bT + c - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2}\right) + e^{\lambda(T-t_z)}\left(2aT + b - \frac{2a}{\lambda}\right)\right) + C_3\left(e^{\lambda(T-t_z)}\left(aT^2 + bT + c - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2}\right) + e^{\lambda(T-t_z)}\left(2aT + b - \frac{2a}{\lambda}\right)\right)\Bigg|_{(t_z^*,T^*).} = \left.\frac{\partial^2 TC_{11}}{\partial t_z \partial T}\right|_{(t_z^*,T^*).}$$

Theorem 1: The cost function in equation (15) is a convex function

Proof: Equation (20) – (23) confirm that the required Hessian matrix is positive definite, hence TC_{11} is convex.

The necessary conditions to obtain a minimum are $\frac{\partial TC_{12}}{\partial t_z} = 0$ and $\frac{\partial TC_{12}}{\partial T} = 0$.

Using (17), by differentiating partially with respect to t_z and T respectively and setting the derivatives to zero.

$$\frac{\partial TC_{12}}{\partial t_z} = -\frac{1}{T}\left\{\frac{h_r}{\mu}\left(at_z^2 + bt_z + c + e^{\mu t_z}\left(at_z^2 + bt_z + c - \frac{(2at_z+b)}{\mu} + \frac{2a}{\mu^2}\right) + \frac{e^{\mu t_z}}{\mu}\left(2at_z + b - \frac{2a}{\mu}\right)\right) + \left\{Zh_0 e^{-\lambda t_z}\right\} + \frac{h_0}{\lambda}\left\{at_z^2 + bt_z + c - \frac{(2at_z+b)}{\lambda} + \frac{2a}{\lambda^2} - e^{\lambda(T-t_z)}\left(aT^2 + bT + c - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2}\right)\right\} + C_3\left(at_z^2 + bt_z + c + e^{\mu t_z}\left(at_z^2 + bt_z + c - \frac{(2at_z+b)}{\mu} + \frac{2a}{\mu^2}\right) + \frac{e^{\mu t_z}}{\mu}\left(2at_z + b - \frac{2a}{\mu}\right)\right) + \left\{ZC_3\lambda e^{-\lambda t_z}\right\} + C_3\left\{at_z^2 + bt_z + c - \frac{(2at_z+b)}{\lambda} + \frac{2a}{\lambda^2} - e^{\lambda(T-t_z)}\left(aT^2 + bT + c - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2}\right)\right\}\right\} = 0 \quad (24)$$

Differentiating equation (17) w.r.t T and setting the result to zero

$$\frac{\partial TC_{12}}{\partial T} = -\frac{1}{T}\left\{\frac{h_o}{\lambda}\left(aT^2 + bT + c - \frac{2a}{\lambda} + \frac{2a}{\lambda^2} - e^{\lambda(T-t_z)}\left(aT^2 + bT + c - \left(\frac{2aT+b}{\lambda}\right) + \frac{2a}{\lambda^2} + \frac{e^{\lambda(T-t_z)}}{\lambda}\left(2aT + b - \frac{2a}{\lambda}\right)\right)\right) + C_3\left(aT^2 + bT + c - \frac{2a}{\lambda} + \frac{2a}{\lambda^2} - e^{\lambda(T-t_z)}\left(aT^2 + bT + c - \left(\frac{2aT+b}{\lambda}\right) + \frac{2a}{\lambda^2} + \frac{e^{\lambda(T-t_z)}}{\lambda}\left(2aT + b - \frac{2a}{\lambda}\right)\right)\right) - TC_{12}\right\} = 0 \quad (25)$$

Solving the nonlinear equations (24) and (25) say (t_z^{2*}, T_2^*). would give the values of t_z^* and T^*. To confirm they exist and unique, we show that $\frac{\partial^2 TC_{12}}{\partial t_z^2}\frac{\partial^2 TC_{12}}{\partial T^2} - \frac{\partial^2 TC_{12}}{\partial t_z \partial T}\frac{\partial^2 TC_{12}}{\partial T \partial t_z} > 0$ at (t_z^{2*}, T_2^*). Therefore, differentiating equation (24) w.r.t t_z, we have

$$\frac{\partial^2 TC_{12}}{\partial t_z^2} = \frac{1}{T}\left\{\frac{h_r}{\mu}\left(2at_z + b + \mu e^{\mu t_z}\left(at_z^2 + bt_z + c - \frac{(2at_z+b)}{\mu} + \frac{2a}{\mu^2}\right) - 2e^{\mu t_z}\left(2at_z + b - \frac{2a}{\mu}\right) - 2a\frac{e^{\mu t_z}}{\mu}\right) + \left\{Zh_0\lambda e^{-\lambda t_z}\right\} + \frac{h_0}{\lambda}\left\{2at_z + b - \frac{2a}{\lambda} + \lambda e^{\lambda(T-t_z)}\left(aT^2 + bT + c - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2}\right)\right\} + C_3\left(2at_z + b - \mu e^{\mu t_z}\left(at_z^2 + bt_z + c - \frac{(2at_z+b)}{\mu} + \frac{2a}{\mu^2}\right) - 2e^{\mu t_z}\left(2at_z + b - \frac{2a}{\mu}\right) - 2a\frac{e^{\mu t_z}}{\mu}\right) - \left\{ZC_3\lambda^2 e^{-\lambda t_z}\right\} + C_3\left\{2at_z + b - \frac{2a}{\lambda} + \lambda e^{\lambda(T-t_z)}\left(aT^2 + bT + c - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2}\right)\right\}\right\} \quad (26)$$

Differentiating equation (25) w.r.t T, we have

$$\frac{\partial^2 TC_{12}}{\partial T^2} = \frac{1}{T}\left\{\frac{h_0}{\lambda}\left\{\left(2aT + b - \lambda e^{\lambda(T-t_z)}\left(aT^2 + bT + c - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2}\right) - e^{\lambda(T-t_z)}(2aT+b) + e^{\lambda(T-t_z)}(2aT+b) + \frac{2ae^{\lambda(T-t_z)}}{\lambda}\right)\right\} + C_3\left\{\left(2aT + b - \lambda e^{\lambda(T-t_z)}\left(aT^2 + bT + c - \frac{(2aT+b)}{\lambda} + \frac{2a}{\lambda^2}\right) - e^{\lambda(T-t_z)}(2aT+b) + e^{\lambda(T-t_z)}(2aT+b) + \frac{2ae^{\lambda(T-t_z)}}{\lambda}\right)\right\} - 2\frac{\partial TC_{12}}{\partial T}\right\} \quad (27)$$

By theorem 1:

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Since $\left.\frac{\partial TC_{12}}{\partial T}\right|_{(t_z^{2*},T_2^*).} = 0$. Thus, value of $\left.\frac{\partial^2 TC_{12}}{\partial T^2}\right|_{(t_z^{2*},T_2^*).} > 0$,

Differentiating the left hand side of equation (25) w.r.t t_z, we have

$$\frac{\partial^2 TC_{12}}{\partial T \partial t_z} = -\frac{1}{T}\left\{h_o\left(e^{\lambda(T-t_z)}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)+e^{\lambda(T-t_z)}\left(2aT+b-\frac{2a}{\lambda}\right)\right)+C_3\left(e^{\lambda(T-t_z)}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)+e^{\lambda(T-t_z)}\left(2aT+b-\frac{2a}{\lambda}\right)\right)+\frac{\partial TC_{12}}{\partial t_z}\right\} \quad (28)$$

Also, differentiating the left hand side of equation (24) w.r.t T, we have

$$\frac{\partial^2 TC_{12}}{\partial t_z \partial T} = -\frac{1}{T}\left\{h_o\left(e^{\lambda(T-t_z)}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)+e^{\lambda(T-t_z)}\left(2aT+b-\frac{2a}{\lambda}\right)\right)+C_3\left(e^{\lambda(T-t_z)}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)+e^{\lambda(T-t_z)}\left(2aT+b-\frac{2a}{\lambda}\right)\right)+\frac{\partial TC_{12}}{\partial t_z}\right\} \quad (29)$$

Evaluating (28) and (29) at the point (t_z^{2*}, T_2^*) and noting that $\left.\frac{\partial TC_{12}}{\partial T}\right|_{(t_z^{2*},T_2^*).} = 0$

$$\left.\frac{\partial^2 TC_{12}}{\partial T \partial t_z}\right|_{((t_z^{2*},T_2^*)).} = -\frac{1}{T}h_o\left(e^{\lambda(T-t_z)}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)+e^{\lambda(T-t_z)}\left(2aT+b-\frac{2a}{\lambda}\right)\right)+ C_3\left(e^{\lambda(T-t_z)}\left(aT^2+bT+c-\frac{(2aT+b)}{\lambda}+\frac{2a}{\lambda^2}\right)+e^{\lambda(T-t_z)}\left(2aT+b-\frac{2a}{\lambda}\right)\right)\Bigg|_{(t_z^{2*},T_2^*).} = \left.\frac{\partial^2 TC_{12}}{\partial t_z \partial T}\right|_{(t_z^{2*},T_2^*).}$$

Theorem 2: The cost function in equation (17) is a convex function

Proof: The determinant of equation (26) – (29), the two principal minors of the Hessian matrix shows positive, it means that the Hessian matrix is positive definite, hence TC_{12} is convex.

Numerical Example

Given an inventory system with the following parameters: Aliyu (2022) $Z = 1000/order$ $N = 0.07$ $c_1 = ₦1500/order$ $c_2 = ₦1/unit\ item$ $c_3 = ₦10/unit\ item$ $l_e = 0.4$ $\mu = 0.1$ $\lambda = 0.1$ $h_0 = ₦3/unit\ item$ $h_r = ₦3/unit\ item$ $a = 3.5$ $b = 1$ $c = 4$

It is observed from table 1 that the stock is exhausted in RW at t_z^* =174 (1.2849) days and the entire inventory cycle become empty at $T^* = 202$ (5.0734) days. The optimal total cost is $TC^* = ₦4095.0842$. it can also be noticed that for both scenarios 1 and 2, the goods finished almost the same period indicating that there it is significantly immaterial whether retailer accept partial credit payment or full credit payment later from the customers.

Table 1: Result of the Numerical examples for the two scenarios

Scenario	t_z	T	TC (₦)
Scenario 1	1.2849	5.0734	4095.0842
Scenario 2	1.2850	5.0739	4094.3254

Sensitivity Analysis

Considering the capacity of the OW, *Z*, the purchasing cost per unit item, c_1, and the selling price per unit, c_2, as the most important parameters and using the results obtained from the above example, we have studied the effect of the parameter's changes on the optimal values of the t_z^*, T^*, and TC^* by varying the three assumed important parameter's values by 10% and 5% up and -10% and -5% below their initial values. The result of the sensitivity analysis is shown and presented in Table 2

Table 2: Sensitivity analysis for scenario 1 and percentage change in the system parameters

	% Change in Parameter	t_z	T	TC
Z	10	1.2517	5.2658	3905.8047
	5	1.2691	5.1714	3999.8110
	0	1.2849	5.0734	4095.0842
	-5	1.2988	4.9714	4191.7946
	-10	1.3107	4.8648	4290.1488

c_1	10	1.2786	5.0527	4124.7105
	5	1.2817	5.0631	4109.8821
	0	1.2849	5.0734	4095.0842
	-5	1.3023	4.1315	4011.6109
	-10	1.2917	5.0941	4065.5785
c_2	10	1.2163	4.8499	4466.5340
	5	1.2500	4.9589	4278.9647
	0	1.2849	5.0734	4095.0842
	-5	1.3208	5.1938	3915.0800
	-10	1.3579	5.3202	3739.1499

It is observed from table 2 that:

1. As the stocking capacity of OW, Z increases, t_z and total cost TC decreases while T increases. This indicates that larger storage allows bulk purchasing, fewer orders, and lower total cost. Managerially, firms should increase storage capacity to achieve lower inventory costs.
2. As the ordering cost $C_{1,}$ increases, t_z and T decreases while total cost TC increases. This occurs because a higher ordering cost directly increases the cost incurred each time an order placed. Although the retailer shortens the cycle length T and reduces the non-shortage period to manage inventory more efficiently, the increase in ordering cost component dominates the total cost structure. Consequently, even with shorter inventory cycles, the overall total cost TC increases due to the higher cost associated with placing orders

Conclusions

In this research, a mathematical inventory model was developed to determine the optimal replenishment policy for a two-warehouse system under trade credit conditions. The application of the differential equation and matrix analysis provided a rigorous framework for establishing the existence and uniqueness of an optimal solution.

The findings reveal that an increase in the trade credit period leads to an increase in the cycle length and total relevant cost due to the higher holding and deterioration costs. Based on the numerical example provided, (scenario 2) it was found that $t_z^* =$1.2850, $T^* =$5.0739 and $TC^* =$4094.3254 is the optimal policy with optimal values of the total cost and the decision variables. It was also observed that warehouse capacity plays a crucial role in determining inventory decisions, as limited owned capacity necessitates the use of rented system warehouse, thereby increasing total cost. From a managerial perspective, the model provides useful insights for inventory managers in minimizing operational costs while maintaining service levels. The study confirms that appropriate control of trade credit terms and storage decisions can significantly reduce total inventory costs. Overall, the model contributes to inventory theory by extending classical models to more realistic two-warehouse systems.

Based on the findings of this study, the following recommendations are made:

1. Inventory managers should carefully select trade credit periods, as longer credit periods tend to increase cycle length and total relevant cost.
2. Organizations operating multiple warehouses should optimize the use of owned warehouse capacity before relying on rented warehouses to minimize holding and deterioration costs.
3. Since some commodities demand response well to the price and/or its availability, the model developed can be extended by considering demand as a function of price, and/or stock level to enhance its practical applicability. Similarly, assuming deterioration rate to be constant may hinder the practicality of the model, therefore, order dependent deterioration rate can be considered to improve the model in the future.

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